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MECHANICAL RESPONSE OF METALLIC, FIBER-REINFORCED BEAMS TO TRANSVERSE IMPACTS

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Abstract—Composite metal beams consisting of steel piano wires embedded in a lead–tin alloy matrix were mounted as cantilevers, and their tips were hit by travelling "hammers". The durations of the impacts were many times the time of transit of a plastic shear wave along the length of the cantilever. It was found that in contrast to the behavior of cantilevers of isotropic metals, the plastic deformation observed was in the form of shear in the clamped section of the cantilever. To study this phenomenon further, long composite beams were mounted so that their central portions were lightly held in a clamp, and two cantilever sections protruded from each end of the clamp. The tip of one of the cantilever sections was hit by a travelling hammer. The plastic deformation observed after the impact was confined to the shearing of the section held between the clamps; the two cantilever sections simply rotated about their fixed ends. The experiments were analyzed by a simplified elastic–plastic technique, and it is shown that there is good agreement between the experimental observations and the theoretical predictions.

SYMBOL NOTATION

A, A' cross-sectional area of beam
b length of section of beam held in clamp
$f_n(x)$ shape factor for nth mode of elastic oscillation
l length of cantilever sections
$M, M(\ )$ moment acting on beam
Q, Q_p transverse force
Q_0 transverse yield force
Q_1 strain hardening parameter
u displacement in x-direction
v displacement in y-direction
γ angle of shear
θ angle of rotation
ρ density
ω angular frequency

INTRODUCTION

We have discussed in earlier papers the mechanical response of composite beams to transverse dynamic loading. The elastic response of fiber-reinforced beams was described in the first paper (Kolsky and Mosquera[1]); it was shown there that the velocity of flexural elastic waves along such beams showed dispersion relations which were in good agreement with those predicted by Sayir[2], both when the elastic anisotropy is small and also when it is very large.

A later paper (Mosquera and Kolsky[3]) described experiments in which metallic beams, which had been axially reinforced by steel piano wires, were deformed plastically as a result of large transverse impacts. These experiments were carried out to see how closely the predictions of the analytical work of Spencer, Jones and their colleagues[4–7] agreed with the experimental observations of impacts designed to simulate the ones that had been considered.

The experiments were carried out on cantilevers, which were of a lead–tin alloy axially reinforced by steel piano wires embedded in it. Large transverse dynamic impacts were produced, either by detonating small explosive charges which accelerated small steel pellets that then hit the tips of the cantilevers, or by hitting the tips with fast-travelling aluminum hammers in a Hyge shock testing machine.

The analyses of Spencer, Jones and their colleagues were all based on the material being modelled as an *ideal fiber-reinforced solid*. For such materials it is assumed that the fibers are completely inextensible, the matrix metal is completely incompressible and its elastic shear modulus is so much greater than its shear strain hardening coefficient that the composite can be modelled as rigid plastic with linear strain hardening. Applied to the materials which we were testing, all these assumptions are only approximate. Nevertheless, it was found that the theoretical predictions were closely confirmed by the experimental results when the duration of the impact was less than the time of transit of a plastic shear wave along the length of the cantilever.

The theoretical treatment predicts that a plastic shear wave will travel along the beam at a velocity which is equal to the square root of the ratio of the strain hardening coefficient to the density of the beam. For the composite we were testing, the strain hardening was approximately linear, so that the velocity of the plastic wave front was effectively independent of the stress level.

The analysis gives an expression for the distance the plastic wave front would be expected to travel for impacts of different magnitudes so long as the duration is less than the time of transit of the whole cantilever. It also gives an expression for the final plastic transverse displacement of the tip of the cantilever after the impact. In the tests using explosive charges the impact durations were sufficiently short, and it was found that both the final position of the plastic front, which could be seen in the impacted specimens, and the displacement of the tip were in good agreement with the theoretical predictions.

For impacts of somewhat longer duration, where the plastic wave front has time to reach the clamped end of the cantilever, the analytical treatment assumes that the wave will be reflected at the clamp, and return towards the cantilever tip. Analytic predictions of the distance it will travel so long as it does not reach the tip, and of the displacement of the tip under these conditions are given in Spencer's treatment. It was found that when the impacts were produced by light hammers in the Hyge machine, such short durations of impact could be achieved. The ciné records of the impact then clearly showed a plastic wave front travelling from the tip to the clamp, but no reflected wave front was visible; when the cantilevers were examined after the tests, no discontinuity could be observed at the location where the plastic wave front should have stopped. This is attributed to the fact that the experimental clamping conditions did not conform to those postulated in the analysis, so that sharp reflections did not take place. However, for such impacts it was found that the observed deflections of the cantilever tips were still in reasonable agreement with the analytical predictions.

For impacts produced in the Hyge testing machine using hammers of very much larger mass, so that the impact durations correspond to many times the transit time of a plastic wave along the cantilever length, it was found experimentally that the distribution of plastic strain in the cantilever specimen after impact was quite different. No plastic deformation could be observed along the length of the cantilever. Plastic strains were entirely confined to the region which was held in the clamp. The purpose of this paper is to elucidate this point further.

The experimental arrangement which was used was to mount long beams so that a short central section, B, is held in a clamp lightly (the reason for light clamping was to minimize frictional forces in the clamp). Two cantilever sections of equal length, A and C, protruded from the two ends of the clamp. The setup is shown diagrammatically in Fig. 1, where the x-axis is vertical, the origin is the point 0 at the center of the boundary between sections A and B, and the y-axis is in the direction in which the blow, $P(t)$, is applied to the tip of cantilever A. The lengths of cantilevers A and C are each taken to be l, while the length of section B is b.

When the beam was made of aluminum and hit in the Hyge testing machine, a plastic hinge was formed near the origin 0; cantilever A was rotated while cantilever C remained undeformed. Figure 2(b) shows such an aluminum specimen after impact. If instead of using an isotropic aluminum beam, a lead–tin alloy beam reinforced with steel piano wires was employed, the final plastic strains were quite different. Thus, here instead of a localized plastic hinge at the origin 0, the whole of section B of the beam undergoes plastic shear, and cantilever sections A and C which show no plastic

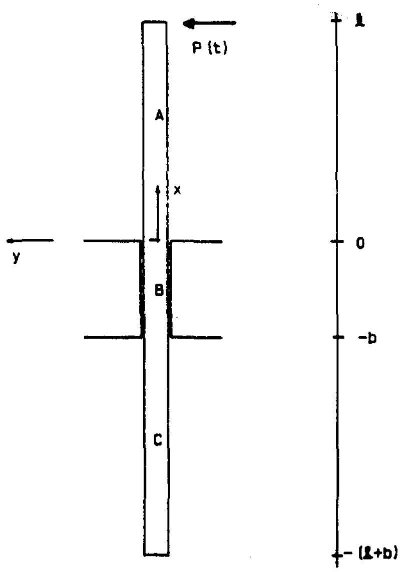
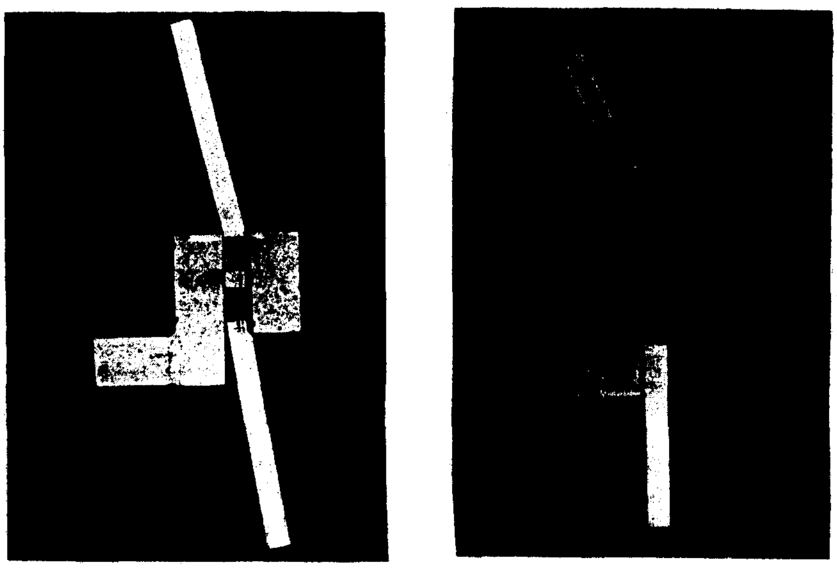


Fig. 1. Arrangement of beam in clamp.

strain are rotated in the same direction about their clamped ends. Such an impacted beam is shown in Fig. 2(a).

In carrying out the experiments in the Hyge testing machine the deformation of the beams was monitored in a number of ways. First, a high-speed cinematographic record was made with a Fastax high-speed ciné camera. Secondly, the force–time relation was recorded by observing the outputs of resistance strain gages mounted on the nose of the hammer. The hammer travelled on a carriage which ran down the guiding rails of the Hyge machine. In addition, the velocity of impact of the hammer was determined by arranging for it to make two electrical contacts a known distance apart just before impact.

Figure 3 gives 72 frames of a ciné record obtained with the Fastax camera, which was running at 1370 frames/sec, of an impact of a hammer of 13.75 lb mass. The



(a) (b)

Fig. 2. Beams after impact: (a) fiber-reinforced beams; (b) isotropic beam.

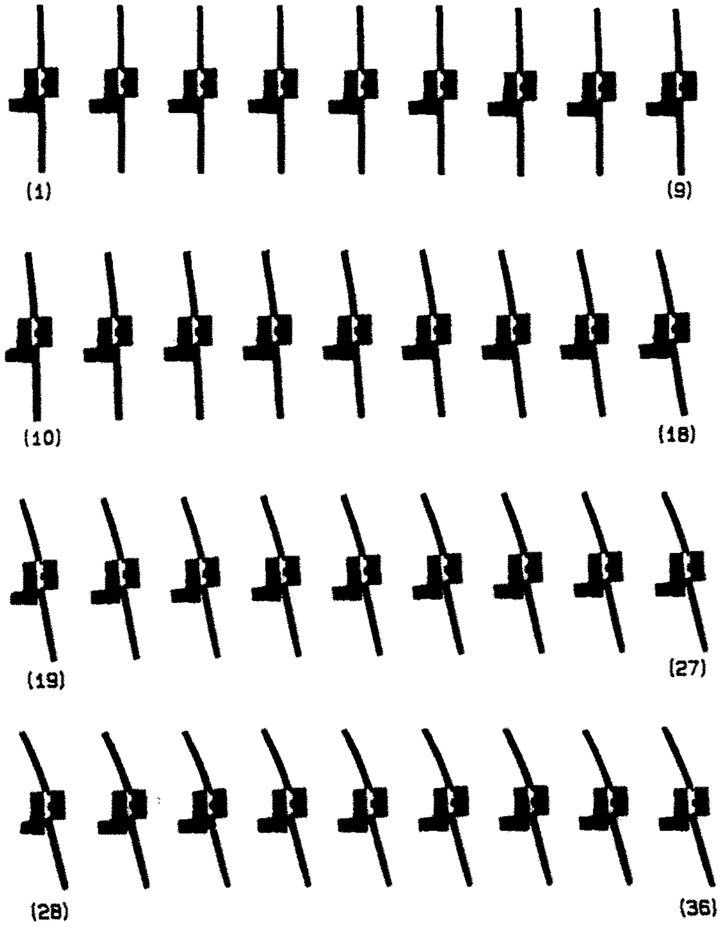


Fig. 3. Ciné record of impact on fiber-reinforced beam. Velocity of impact 180 in/sec; speed of camera 1370 frames/sec.

hammer hit the tip of cantilever A at a speed of 180 in/sec. Enlargements of four representative frames are shown in Fig. 4. Figure 4(a) shows the beam before impact has commenced; Fig. 4(b), which is the 11th frame, shows the specimen 8.03 msec after impact has commenced. It corresponds to the time when plastic yield is just beginning in section B. Figure 4(c) is the 57th frame (41.6 msec after impact) and is the time when the deflection of the tip of A is a maximum, the cantilever A is here bent elastically and has rotated about the origin 0. Cantilever C is unstrained but has rotated about the center point of its boundary with B; section B is sheared. Finally, Fig. 4(d) shows the residual plastic deformation when the impact is over, and the elastic strains have disappeared.

Figure 5 shows the stress–time records obtained from the resistance strain gages for the four different impacts, with the carriage travelling at the following velocities in inches per second, (a) 33.3, (b) 89.7, (c) 120.6, and (d) 156.4.

ANALYSIS OF THE IMPACT

During the impact both elastic and plastic deformations occur, and a complete analysis of the problem which simultaneously takes into account all the effects is certain to be complicated and will probably prove intractable except insofar as obtaining numerical solutions for specific problems with the aid of a sufficiently sophisticated computer program. We propose here instead to use the simplified elastic–plastic (S.E.P) technique developed by Symonds[8]. We have used this technique for solving structural problems (Symonds, Kolsky and Mosquera[9]) and shown that its predictions agree well both with the predictions of a much more complex computer program based on

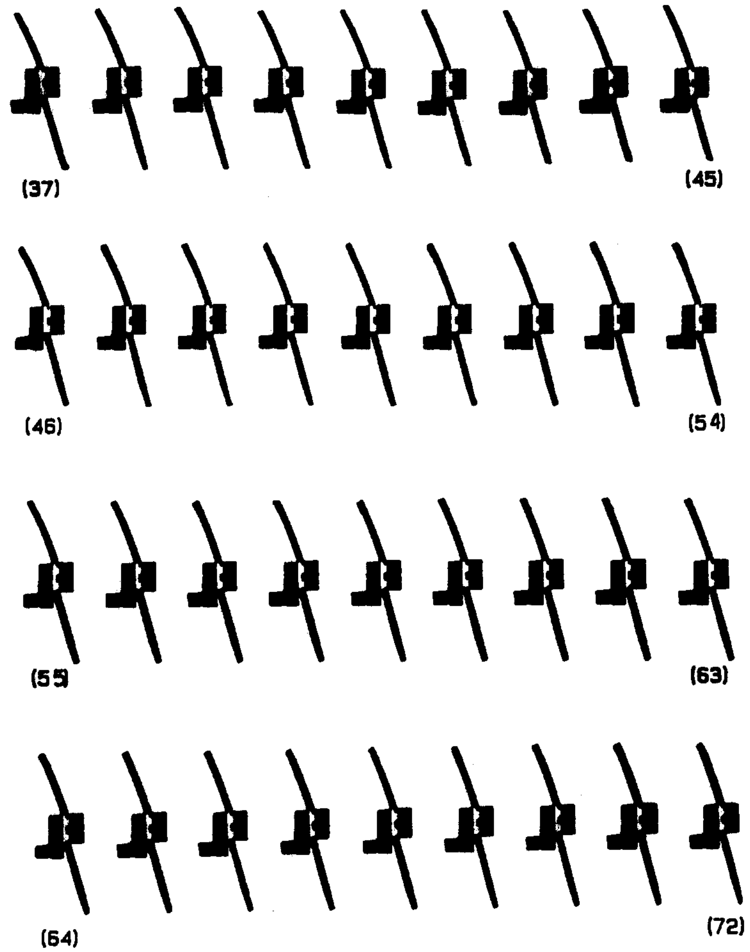


Fig. 3. Continued.

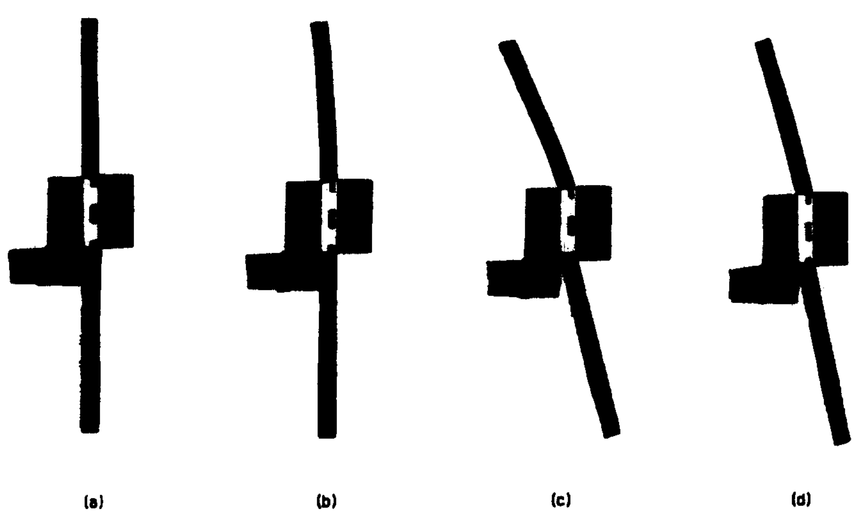


Fig. 4. Four selected frames from ciné record: (a) beam before impact; (b) beam 8.03 msec after commencement of impact; (c) beam 41.6 msec after commencement of impact; (d) beam after impact.

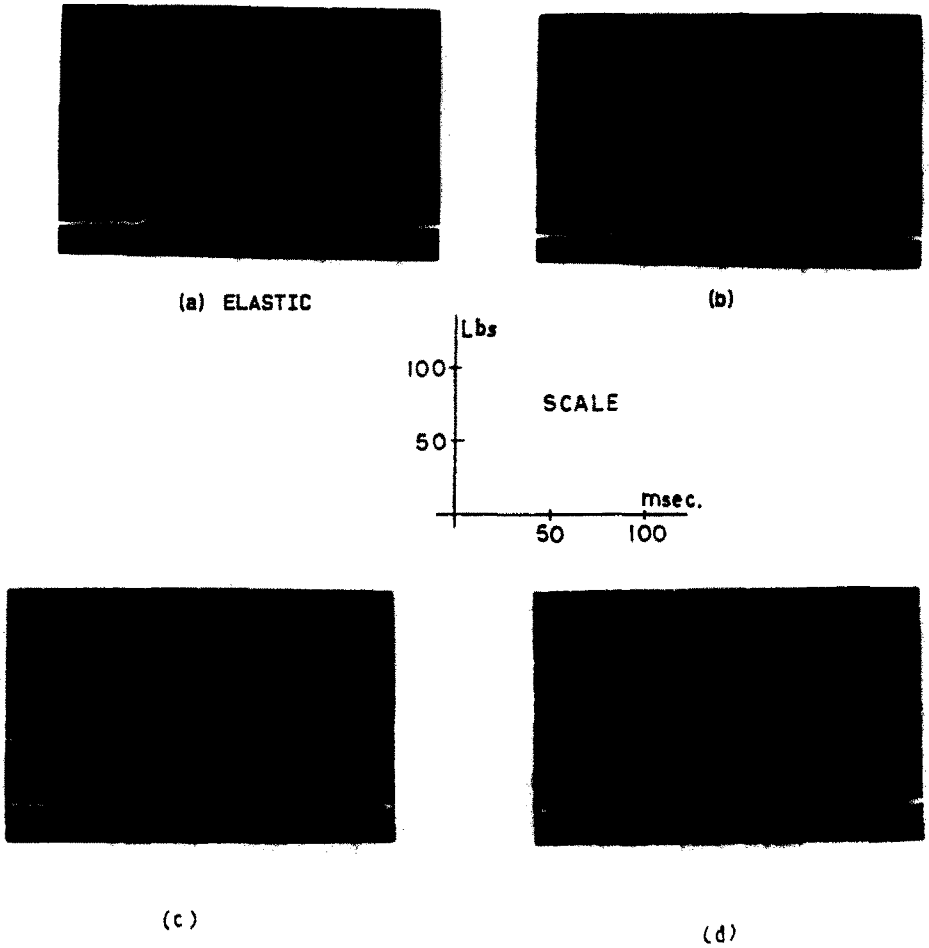


Fig. 5. Force–time records for four impacts. Velocity of hammer in in/sec: (a) 33.2; (b) 89.2; (c) 120.6; (d) 156.4. (Impact (a) produced no plastic yield.)

finite element methods (ABAQUS) and with the experimental observations of impacts on model frames.

This technique is based on dividing up the history of the deformation into three distinct stages. The first stage is the elastic stage where the problem is dealt with by the standard methods of elasticity. The second stage is one where the response of the structure is treated as rigid plastic, and further elastic deformations are ignored. The final stage is once again elastic and corresponds to elastic recovery and possible elastic oscillations.

In the problem we are considering—namely the response of a long beam, the center section of which is lightly clamped—the first stage of the deformation corresponds to the bending of the cantilever A by an imposed force $P(t)$ acting at its tip (cf. Fig. 1). This elastic problem is an old one and can be treated in terms of the normal flexural modes of vibration as formulated by St. Venant[10].

The treatment we use here is similar to that described by Warburton[4] in his book. The equation of motion of a beam subjected to a distributed applied force $P(t)$ acting in the y-direction on a beam lying in the x-direction can be written as

$$\rho A \frac{\partial^2 v}{\partial t^2} + EI \frac{\partial^4 v}{\partial x^4} = S(x)P(t), \tag{1}$$

where ρ is the density of the beam, A is its cross-section, E is its Young's modulus in the axial direction, I is the moment of inertia of the cross-sectional area about the neutral axis and $S(x)$ defines the spatial distribution of the applied force. Thus, if the force is applied at the tip of cantilever A, $S(x) = \delta(x - l)$.

Now we look for a solution in the form of a sum of the normal modes of vibration of the beam. Thus, for the nth mode we have

$$v_n = f_n(x)g_n(t) \qquad (f_n(x) \text{ is the shape factor for the } n\text{th mode}).$$

The complete solution is then

$$v(x, t) = \sum_n f_n(x)g_n(t). \tag{2}$$

By using the orthonormal properties of the modes, we can finally derive the expression

$$v(x, t) = \sum_n \frac{f_n(x)f_n(l)}{\omega_n^2 \rho A l} \int_0^t P(\tau) \sin \omega_n(t - \tau)\, \mathrm{d}\tau. \tag{3}$$

Here ω_n is the natural frequency of the nth mode of oscillation of the cantilever.

Expression (3) can be readily computed when $P(t)$ is known.

Now in the second stage of the S.E.P. treatment the structure is treated as behaving in a rigid–plastic manner, and no further elastic deformations are assumed to take place.

In our specific problem, so long as $b < l$, the plastic deformation takes place in section B (we can see this is so by considering the quasi-static loading of the system when the transverse force that the clamp exerts on section B at the boundary with cantilever C is lP/b in order that moments balance about 0). The plastic shear in section B results in rotation of both cantilevers A and C, which rotate about their junctions with B, but which undergo no plastic deformation along their lengths. Since the material is assumed to be rigid–plastic, the shear force has its yield value Q_p so long as plastic flow continues.

Now in this stage it is not unreasonable to treat section B as an ideal fiber-reinforced rigid–plastic solid: the material is assumed to be inextensible in the fiber direction, i.e. $\partial u/\partial x = 0$, where u is the displacement in the x-direction. Thus since we assume no variation with the z-co-ordinate, we have

$$u = u(y, t). \tag{4}$$

Further, since ideal fiber-reinforced materials are assumed to be incompressible, $\partial v/\partial y = 0$, so that

$$v = v(x, t). \tag{5}$$

The assumptions which are made for a rigid–plastic material are that if the shear force is Q and the yield force is Q_p, then

$$-Q_p \le Q \le Q_p. \tag{6}$$

Thus, the transverse force must lie between the limits $-Q_p$ and Q_p. When it assumes either of these values, plastic flow takes place, the direction depending on whether the shear force has its positive or negative limiting value. This fact can be expressed as

$$(Q - Q_p)\dot{\gamma} \ge 0, \qquad (Q + Q_p)\,\dot{\gamma} \ge 0,$$

where $\dot{\gamma}$ is the shear strain rate.

Now we consider the deformation of section B of the beam. There is some motion in this section, but the inertia terms are small. There are no displacements in the y-

direction (i.e. $v = 0$), and the maximum displacement in the x-direction u is given by $(h \sin \theta')/2$, where θ' is the maximum angle of shear.

Now since section B is not rigidly clamped it is assumed that no large axial forces will be exerted on it by the clamp; there will, however, be large normal stresses exerted on the two faces of the beam. Since during plastic flow the shear strain is constant along the clamp section, the forces exerted on this section must be concentrated at the two ends: $x = 0$ and $x = -b$.

Since we are neglecting D'Alembert forces in section B, the equation of equilibrium may be written as

$$dM/dx + Q(x) = 0, \tag{7}$$

where Q is the transverse force acting on the section of the beam at x, and $M(\dot{x})$ is the moment acting on this section. M results from axial forces acting on the beam, and in an ideal fiber-reinforced solid these are confined to boundary layers of the beam.

Now if we integrate eqn (7) between the limits $x = -b$ and $x = 0$, we get

$$M(0) - M(-b) + Q(0)x0 - Q(-b)x(-b) = 0.$$

Thus,

$$M(0) - M(-b) + Q_b b = 0. \tag{8}$$

Now, let us consider the equations of rotational motion of the cantilevers A and C. In the plastic stage they both rotate through the same angle θ, where if the displacement of the tip of A is v_0, $\theta = v_0/l$. Now for cantilever A we get

$$P(t)l + M(0) = \tfrac{1}{3}\rho A' l^3 \ddot{\theta} \tag{9}$$

(where A' is the cross-sectional area of the beam), whereas the similar equation for cantilever C is

$$-M(-b) = \tfrac{1}{3}\rho A' l^3 \ddot{\theta}. \tag{10}$$

When we add (9) and (10), we obtain

$$P(t)l + M(0) - M(-b) = \tfrac{2}{3}\rho A' l^3 \ddot{\theta}. \tag{11}$$

From eqn (8), $M(0) - M(-b) = -Q_p b$, so that (11) becomes

$$P(t)l - Q_p b = \tfrac{2}{3}\rho A' l^3 \ddot{\theta} = \tfrac{2}{3}\rho A' l^2 \ddot{v}_0. \tag{12}$$

In order to find the value of the maximum displacement v_t we must integrate eqn (12) numerically with the appropriate initial conditions.

Now, in an earlier investigation (Mosquera and Kolsky[3]) it was shown that the lead–tin composite could be adequately modelled as rigid–plastic with linear strain hardening; thus,

$$Q_p = Q_0 + Q_1 \gamma.$$

Q_1 was the linear strain hardening coefficient, and Q_0 was the yield force, which was found to depend on the strain rate $\dot{\gamma}$.

Now in analyzing the problem that we are considering here, we are faced with the fact that we do not know *a priori* either γ or $\dot{\gamma}$ during the impact and that we need values of Q_p in order to find them.

In an attempt to do this, we have used an iterative procedure, making the approximation that

$$Q_{1p} = Q_0(\dot{\gamma}) + \tfrac{1}{2}Q_1\gamma_s, \qquad (13)$$

where γ_s is the value of the shear strain at the end of plastic flow. We do not, of course, know this, but hope to find it by a sufficient number of numerical iterations.

In our earlier paper (Mosquera and Kolsky[3]), we showed that $Q_0(\dot{\gamma})$ could be reasonably represented by an equation of the type first suggested by Cowper and Symonds[12]:

$$Q_0(\dot{\gamma}) = Q_0(1 + (\dot{\gamma}/\dot{\gamma}_0)^{1/n}). \qquad (14)$$

Here Q_0 is the limiting value of $Q_0(\dot{\gamma})$ as $\dot{\gamma} \to 0$, while $\dot{\gamma}_0$ is the value of $\dot{\gamma}$ when $Q_0(\gamma) = 2Q_0$. n is a numerical constant, which for this particular composite gave the best agreement with observations when $n = 5.5$.

In tackling the problem, the numerical procedure that we adopted was to divide the loading history $P(t)$ into a number of small steps (each step was chosen to be of 0.5-msec duration). We then made our first iteration by assuming that $\gamma_s = 0$ and $\dot{\gamma}$ for the first step was equal to $\dot{\gamma}_e$, the elastic strain rate at the time t_c where we assumed the rigid–plastic stage began. For each subsequent step we used the ratios $\Delta\gamma_{n-1}/\Delta t$ as a reasonable approximation for $\dot{\gamma}_n$, where $\dot{\gamma}_n$ is the value of $\dot{\gamma}$ in the nth time interval.

This calculation gave us our first value, $[\gamma_s]_1$. We now proceeded to the second iteration using the relation

$$Q_p = Q_0(1 + (\dot{\gamma}/\dot{\gamma}_0)^{1/n}) + \tfrac{1}{2}[\gamma_s]_1. \qquad (15)$$

This in turn gave us a second value for γ_s which could be denoted by $[\gamma_s]_2$. We found that after four or five iterations the value of γ_s assumed a stable value.

Finally, we had to establish a method of choosing a time t_c at which to go from the elastic stage of the analysis to the rigid–plastic one. We must also find a value for v_0^*, the velocity for the tip at time t_c. We have done this by the use of a maximum work principle as described in [9], using the "minimum Δ_0" device, which was first put forward by Martin and Symonds[13].

RESULTS AND CONCLUSIONS

A series of seven experiments were carried out, and the velocities of impact ranged from 49.3 in/sec to 180.3 in/sec. For each experiment force–time records were obtained, and the maximum displacements of the tips of cantilevers A and C were measured on the ciné film records. After each experiment the permanent displacements of the two cantilever tips were measured, and the results were compared with the values predicted by the S.E.P. analysis. In view of the very approximate nature of this analytic treatment, the agreement found between the experimental observations of the final plastic strain and that predicted by the theory was very good. The observed values are given in Table 1 and are compared with the predicted ones. The comparisons are shown graphically in Fig. 6.

The values of the maximum displacements are also given and are compared with the predicted values. The comparison is shown graphically in Fig. 7. It may be seen that the agreement here is not so good, the experimentally observed values being considerably larger than the predicted ones. These differences are attributed to the assumption in the theoretical elastic analysis of a beam rigidly clamped at its end, and this does not apply to our experimental setup.

Perhaps the main contribution of this work is that a type of yield which has hitherto received very little attention either experimentally or theoretically is described and discussed. The yield which occurs in beams of these fiber-reinforced metal composites

Table 1. Experimental and theoretical values of tip displacement

Test	Velocity of impact (in/sec)	w_{max} (in)	Observations: Permanent displacement w_p Cant A	Cant C	Predicted values: w_{max}	w_p
1	49.3	0.35	0.10	0.07	0.17	0.02
2	65.6	0.50	0.25	0.20	0.29	0.15
3	89.7	0.67	0.38	0.42	0.49	0.35
4	120.6	0.68	0.42	0.38	0.60	0.40
5	131.4	0.80	0.52	0.46	0.72	0.58
6	156.4	0.88	0.64	0.60	0.79	0.66
7	180.3	1.30	0.95	0.91	1.05	0.81

Dimensions

Length of cantilevers A and C	4.5 in.
Length of section B	1.8 in.
Beam thickness	0.5 in.
Beam width	1.0 in.

Mass of hammer 13.75 lb.
Constants of beam composite Q_0 = 91.3 lb, Q_1 = 2250 lb.

is in the form of plastic shear flow over an extended region of the beam. This contrasts with the more localized plastic hinges observed with isotropic metals.

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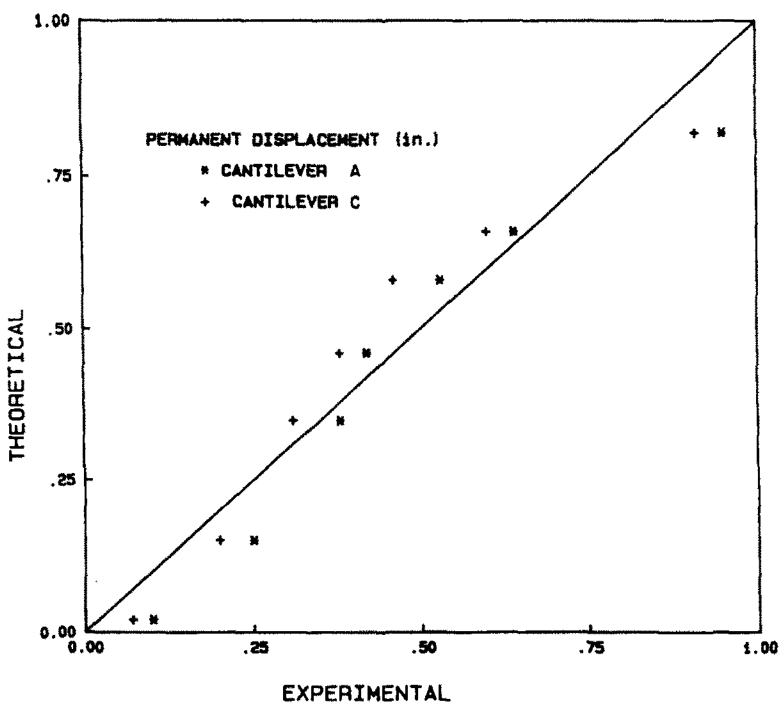

Fig. 6. Comparison of observed and predicted values of final plastic tip displacements.

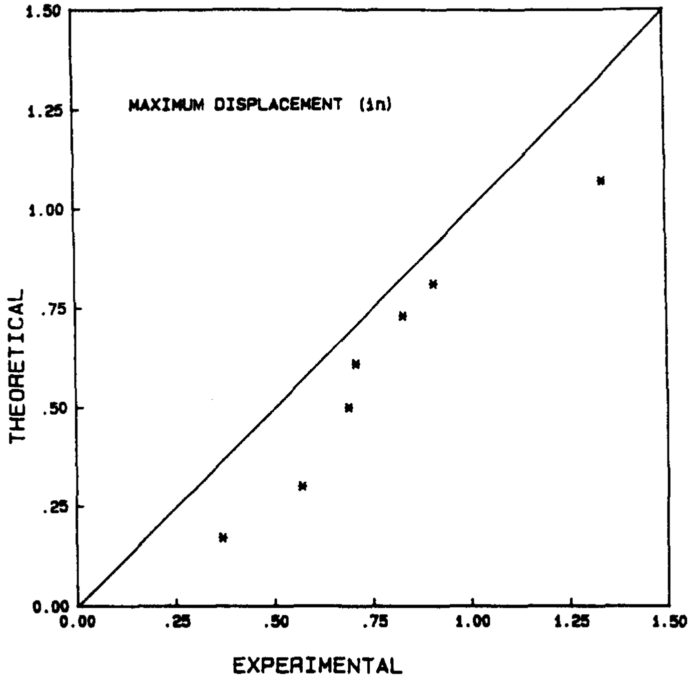

Fig. 7. Comparison of observed maximum displacements with theoretical predictions.

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